Note Title 04/07/2010

Chapter 3 Continuous-Time Stochastic Processes

General Description of a continuous-time stochastic process

A collection of random variables

$\{X_t ; t \geq 0\}$

For each t, X_t is a RV.

t represents time

There is a filtration or information structure

$\{\mathcal{F}_t, t \geq 0\}$,

$\mathcal{F}_t$ represents information from $\{X_t, t \geq 0\}$ up to time t. Thus $\mathcal{F}_s \subset \mathcal{F}_t$, $s \leq t$

may be interpreted as all the paths up to time t.

What is a path?

A senario up to time t.

Example: Let S_t be the stock price at t for $0 \leq t \leq \infty$

A path up to time T is all the stock prices between time 0 and time T.

A special continuous-time stochastic process

The standard (Arithmetic) Brownian motion (BM)
(Wiener process)

p 48

$\{ W_t ; \ t \geq 0 \}$

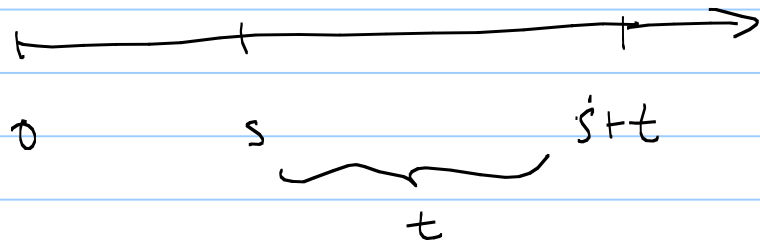
(i) W_t is continuous (has continuous paths)
No jumps in value
$W_0 = 0$

(ii) For a fixed t, $W_t \sim N(0, t)$

$$E(W_t) = 0, \quad Var(W_t) = t$$

(iii) Independent Increments

$$W_{s+t} = \underbrace{W_s}_{\text{current}} + \underbrace{(W_{s+t} - W_s)}_{\text{increment}}$$



$W_{s+t} - W_s$ is independent of W_s

Thus $W_{s+t} - W_s \sim N(0, t)$

$\{ W_t ; t \geq 0 \}$ with (i), (ii) and (iii)

is called the standard BM

Ex 3.1 If $Z \sim N(0,1)$ and $X_t = \sqrt{t}\, Z$

Is X_t a standard BM?

Check (i), (ii) & (iii)

(i) X_t is continuous. $X_0 = 0 Z = 0$.

(ii) $X_t \sim N(\mu, \sigma^2) = N(0, t)$

$E(X_t) = \sqrt{t}\, E(Z) = 0$, $Var(X_t) = (\sqrt{t})^2 Var(Z) = t$

(iii) $X_{s+t} - X_s = \sqrt{t+s}\, Z - \sqrt{s}\, Z$

$$= \left(\sqrt{t+s} - \sqrt{s}\right) Z$$

and $X_s = \sqrt{s}\, Z$

$$\mathrm{Cov}(X_s, X_{s+t} - X_s) = \mathrm{Cov}\left(\sqrt{s}\, Z, (\sqrt{t+s} - \sqrt{s}) Z\right)$$

$$= \sqrt{s}\,(\sqrt{t+s} - \sqrt{s})\, \mathrm{Cov}(Z, Z)$$

$$= \sqrt{s}\left(\sqrt{t+s} - \sqrt{s}\right) \mathrm{Var}(Z) = \sqrt{s}\left(\sqrt{t+s} - \sqrt{s}\right) \neq 0$$

Ex 3.2 W_t and $\tilde{W}_t$ are two independent standard BM's. $-1 < \rho < 1$

Define

$$X_t = \rho W_t + \sqrt{1-\rho^2}\,\tilde{W}_t$$

Is X_t a std BM?

(i) Continuous . $X_0 = \rho W_0 + \sqrt{1-\rho^2}\,\tilde{W}_0 = 0 + 0 = 0$

(ii) $X_t \sim N(\mu, \sigma^2) = N(0, t)$

$$E(X_t) = \rho E(W_t) + \sqrt{1-\rho^2}\, E(\tilde{W}_t) = \rho \times 0 + \sqrt{1-\rho^2}\, 0 = 0$$

$$\mathrm{Var}(X_t) = \mathrm{Var}\left(\rho W_t + \sqrt{1-\rho^2}\,\tilde{W}_t\right)$$

$$= \mathrm{Var}(\rho W_t) + \mathrm{Var}\left(\sqrt{1-\rho^2}\,\tilde{W}_t\right)$$

$$= \rho^2\,\mathrm{Var}(W_t) + (1-\rho^2)\,\mathrm{Var}(\tilde{W}_t)$$

$$= \rho^2 t + (1-\rho^2) t = t$$

(iii) $$X_{s+t} - X_s = \left(\rho W_{s+t} + \sqrt{1-\rho^2}\,\tilde{W}_{s+t}\right)$$

$$-\left(\rho W_s + \sqrt{1-\rho^2}\,\tilde{W}_s\right)$$

$$= \rho\left(W_{s+t} - W_s\right) + \sqrt{1-\rho^2}\left(\tilde{W}_{s+t} - \tilde{W}_s\right)$$

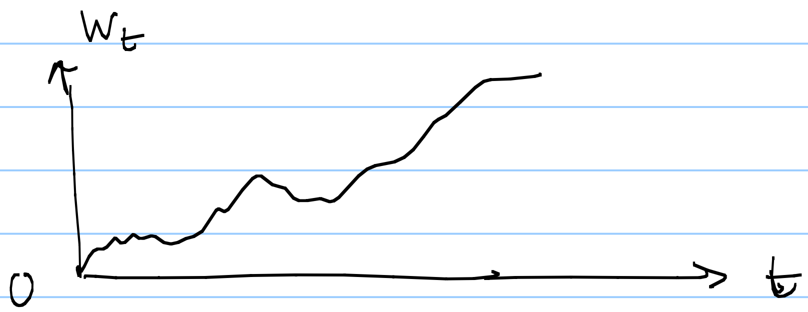
Independent of $X_s = \rho W_s + \sqrt{1-\rho^2}\,\tilde{W}_s$? YES!

X_t is a std BM. But X_t is correlated with W_t

and $\mathrm{cov}(X_t, W_t) = \rho t$

Other properties (P49)

- Each path is not differentiable everywhere



$f(t)$

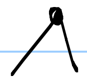
is differentiable

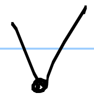
if $f'(t)$

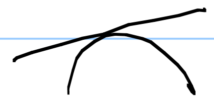
exists

$$\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = f'(t)$$

The path takes sharp turn at any point

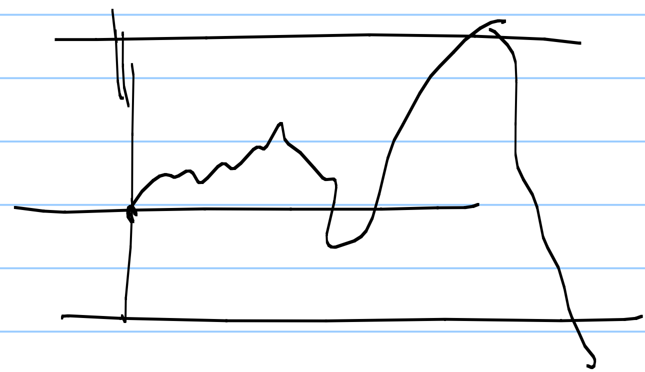




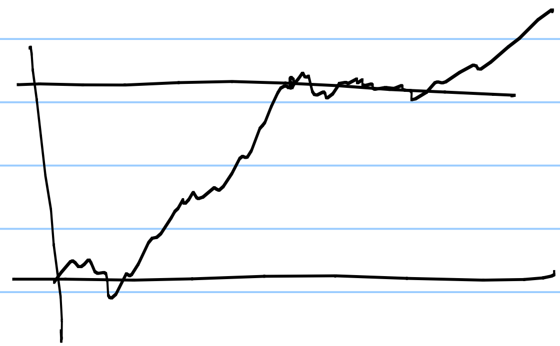


- A path will go everywhere.

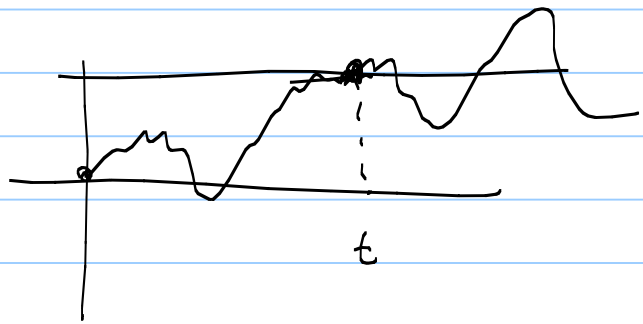
Covering $(-\infty, \infty)$



- A path will oscilate intensively



- A path start at any future point all over again



General BM with drift μ and volatility σ

$X_t \sim$ General BM if

$X_t = x_0 + \mu t + \sigma W_t$

x_0 = starting point

(i) X_t is continuous and $X_0 = x_0$

(ii) $X_t \sim N(x_0 + \mu t, \sigma^2 t)$ because

$E(X_t) = x_0 + \mu t + \sigma E(W_t) = x_0 + \mu t$

$$\mathrm{Var}(X_t) = \mathrm{Var}(\sigma W_t) = \sigma^2 \mathrm{Var}(W_t) = \sigma^2 t.$$

(iii) $X_{s+t} - X_s$ is independent of X_s

(independent increments)

Since $X_{s+t} - X_s = \mu t + \sigma(W_{s+t} - W_s)$,

independent of $x_0 + \mu s + \sigma W_s$

$$X_{s+t} - X_s \sim N(\mu t, \sigma^2 t)$$

P50 BM as stock model?

If it can be used, then

$S_t = S_0 + \mu t + \sigma W_t$

As a result, $S_t < 0$ for any t

because bullet point #2.

Ex 3.3 $P(S_T < 0) > 0$

Proof $P(S_T < 0) = P(S_0 + \mu T + \sigma W_T < 0)$

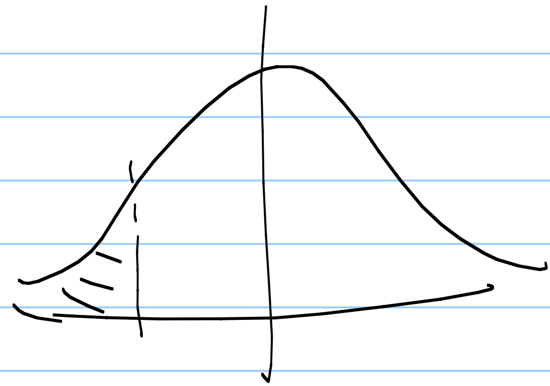
$$= P\left(W_T < \frac{-S_0 - \mu T}{\sigma} \right)$$

$$= P\left(\frac{W_T}{\sqrt{T}} < \frac{-S_0 - \mu T}{\sigma\sqrt{T}} \right)$$

$$= N\left(\frac{-S_0 - \mu T}{\sigma\sqrt{T}} \right) > 0$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{y^2}{2}} \, dy$$

d.f. of $N(0, 1)$



Use the geometric BM (GBM) as stock model

$$S_t = S_0 e^{\mu t + \sigma W_t}$$

Let $x_0 = \ln S_0$. $\quad S_t = e^{x_0 + \mu t + \sigma W_t}$

~~$\left[S_t = S_0 e^{\mu t + \sigma \sqrt{t} Z} \right]$ Wrong!!~~

For fixed t, S_t is lognormal

and $E(S_t) = S_0 e^{(\mu + \frac{1}{2}\sigma^2) t}$

Section 3.2

- Deterministic Calculus p52-54

Differentiation

$f(t)$ is differentiable if

$$\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$ exists and denoted

by $f'(t) = \frac{df}{dt}$

$f(t)$ is called a smooth function

Integration

$$\int_a^b g(t)\,dt = \lim_{n \to \infty} \sum_{i=1}^{n} g\left(a + \frac{i}{n}(b-a)\right) \frac{b-a}{n}$$

Newton's Fundamental Theorem

$$f(b) - f(a) = \int_a^b f'(t)\,dt$$

Ordinary Differential Equations (ODE)

Example

$$a f''(t) + b f'(t) + c f(t) + d = 0, \quad f(0) = x_0$$

$$f'' = \frac{d^2 f}{dt^2}$$

$f(t)$ satisfying the above is called
a solution of the ODE.

Partial Differential Equations (PDE)

$f(t, x)$, more than one variable

It has two partial derivatives

$$\frac{\partial f}{\partial t} \quad \text{and} \quad \frac{\partial f}{\partial x} \quad , \quad \frac{\partial^2 f}{\partial t^2} \quad , \quad \frac{\partial^2 f}{\partial x^2} \quad , \quad \frac{\partial^2 f}{\partial t \partial x}$$

$$\frac{\partial f}{\partial t} + a \frac{\partial^2 f}{\partial x^2} + b \frac{\partial f}{\partial x} + c f = 0 \qquad \text{(PDE)}$$

How to handle W_t ?

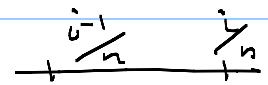
Can we have $\frac{dW_t}{dt}$?

Stochastic calculus starts with integration

- Define stochastic integral with respect to W_t (Ito integral)

For any given continuous-time stochastic process Y_t, $t \geq 0$

Define $$\int_0^a Y_t \, dW_t = \lim_{n \to \infty} \sum_{i=1}^{\frac{a}{n}-1} Y_{\frac{i-1}{n}} \left[W_{\frac{i}{n}} - W_{\frac{i-1}{n}} \right]$$



limit always exits

Properties

$$\int_0^a Y_t \, dW_t \quad \text{is a RV.}$$

The mean

$$E\left\{ \int_0^a Y_t \, dW_t \right\} = 0$$

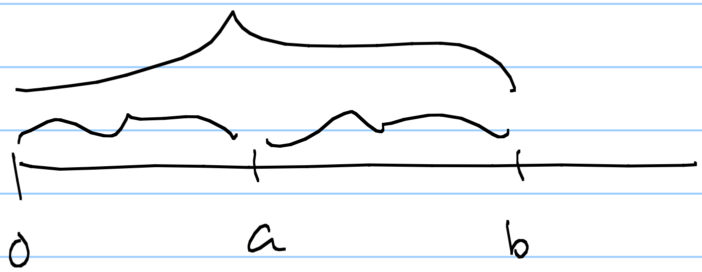
The variance

$$\operatorname{Var}\left\{\int_0^a Y_t\, dW_t\right\} = E\left\{\left[\int_0^a Y_t\, dW_t\right]^2\right\}$$

$$= \int_0^a E(Y_t^2)\, dt$$

$$\int_0^b Y_t\, dW_t = \int_0^a Y_t\, dW_t + \int_a^b Y_t\, dW_t$$

$0 < a < b$



Stochastic Differential Equations (SDE)

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s$$ ← Real meaning

$\mu_t, \sigma_t, t \geq 0$ are two stochastic processes

P56

μ_t = drift process, σ_t = volatility process

Usually, it is written as

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t$$ ← convention

There is an advantage of using the differential form :

many rules apply

X_t is called an Ito process.

If $\mu_t = \mu(t, X_t)$, $\sigma_t = \sigma(t, X_t)$

i.e

$$dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t$$

A differential equation (SDE)

2. $$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

$$\mu_t = \mu X_t \, , \quad \sigma_t = \sigma X_t$$

what is X_t ?? (Later)

X_t is a solution of the SDE with initial value x_0 $\downarrow$

Examples.

① $dX_t = \mu\, dt + \sigma\, dW_t$, $X_0 = x_0$

$$\Rightarrow X_t = x_0 + \int_0^t \mu\, ds + \int_0^t \sigma\, dW_s$$

$$= x_0 + \mu t + \sigma \int_0^t dW_s$$

$$= x_0 + \mu t + \sigma \left[W_t - W_0 \right] = x_0 + \mu t + \sigma W_t$$

Brownian motion with initial value x_0, drift μ and volatility σ !

3.3. Ito's Lemma

Let X_t be a solution of the SDE

$$dX_t = \underset{\mu_t}{\mu(t, X_t)}\, dt + \underset{\sigma_t}{\sigma(t, X_t)}\, dW_t ,$$

and $g(t, x)$ is a deterministic function of t and x.

$Y_t = g(t, X_t)$ is a continuous process

Question: what SDE is Y_t satisfied?

$$dY_t = \left[\frac{\partial}{\partial t} g(t, X_t) + \mu(t, X_t) \frac{\partial}{\partial x} g(t, X_t) + \frac{1}{2}\left[\sigma(t, X_t)\right]^2 \frac{\partial^2}{\partial x^2} g(t, X_t)\right] dt + \sigma(t, X_t) \frac{\partial}{\partial x} g(t, X_t)\, dW_t$$

$\frac{\partial}{\partial t} g(t, x)$ = partial derivative w.r.t. t

$\frac{\partial}{\partial x} g(t, x)$ = partial derivative w.r.t. x.

P59 $\quad g(t, x) = g(x)$

Example $\quad dX_t = \mu X_t \, dt + \sigma X_t \, dW_t \qquad \mu_t = \mu X_t$

$\sigma_t = \sigma X_t$

Let $\quad g(x) = \ln x \quad \Rightarrow \quad Y_t = \ln X_t$

Apply Ito's Lemma

$$g'(x) = \frac{1}{x} \quad , \quad g''(x) = -\frac{1}{x^2}$$

$$dY_t = \Big[\underbrace{\mu X_t \cdot \frac{1}{X_t}}_{\mu_t g'(X_t)} + \underbrace{\frac{1}{2} (\sigma X_t)^2 \left(- \frac{1}{X_t^2} \right)}_{\frac{1}{2} \sigma_t^2 g''(X_t)} \Big] dt$$

$$+ (\sigma X_t) \cdot \frac{1}{X_t} \, dW_t \quad \leftarrow \quad \sigma_t g'(X_t)$$

$$dY_t = \left[\mu - \frac{1}{2}\sigma^2\right] dt + \sigma\, dW_t$$

$$Y_t = Y_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

Since $Y_t = \ln X_t$, $Y_0 = \ln X_0$

$$\ln X_t = \ln X_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

$$\Rightarrow \quad X_t = X_0\, e^{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t} \qquad \text{GBM!}$$

Can express a GBM stock price in two ways:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

OR

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

instantanous volatility

Instantanous expected return

Instantanou's return

$$E(S_t) = S_0 e^{[(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\sigma^2]t}$$

$$= S_0 e^{\mu t}$$

μ = expected return

$\mu - \frac{1}{2}\sigma^2$ is NOT!

Exercise 3.5

$$dX_t = \mu_t X_t \, dt + \sigma_t X_t \, dW_t$$

where μ_t, σ_t are deterministic functions.

Try $Y_t = \ln X_t$

$$dY_t = (\mu_t - \frac{1}{2}\sigma_t^2)\, dt + \sigma_t \, dW_t$$

$$Y_t = Y_0 + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) \, ds + \int_0^t \sigma_s \, dW_t$$

$$\ln X_t = \ln X_0 + \int_0^t [\mu_s - \frac{1}{2}\sigma_s^2) \, ds + \int_0^t \sigma_s \, dW_s$$

$$X_t = X_0 \, e^{\int_0^t [\mu_s - \frac{1}{2}\sigma_s^2] ds + \int_0^t \sigma_s dW_s}$$

X_t is lognormal but not a GBM

This is a stock model to address the volatility smile.

* Derivation of Ito's Lemma

Use the Taylor expansion!

$$d\,g(t,x) =$$

$$g(t+dt,\, x+dx) - g(t,x) =$$

$$\left[\frac{\partial}{\partial t} g(t,x)\right] dt + \left[\frac{\partial}{\partial x} g(t,x)\right] dx$$

$$+ \underbrace{\frac{1}{2}\left[\frac{\partial^2}{\partial t} g(t,x)\right] dt^2}_{0} + \underbrace{\left[\frac{\partial^2}{\partial t \partial x} g(t,x)\right] dt\,dx}_{0} + \frac{1}{2}\left[\frac{\partial^2}{\partial x^2} g(t,x)\right] (dx)^2$$

+ higher order terms

Do the first-order approximation:

$$dY_t = d\,g(t, X_t)$$

$$\approx \left[\frac{\partial}{\partial t} g(t,X_t)\right] dt + \left[\frac{\partial}{\partial x} g(t,X_t)\right] dX_t$$

$$+ \frac{1}{2}\left[\frac{\partial^2}{\partial x^2} g(t, X_t)\right](dX_t)^2$$

$$dY_t = \left[\frac{\partial}{\partial t} g(t, X_t)\right] dt + \left[\frac{\partial}{\partial x} g(t, X_t)\right](\mu_t\, dt + \sigma_t\, dW_t)$$

$$+ \frac{1}{2}\left[\frac{\partial^2}{\partial x^2} g(t, X_t)\right] \sigma_t^2\, dt$$

$$(\mu_t\, dt + \sigma_t\, dW_t)^2 = \mu_t^2 (dt)^2 + 2\mu_t\sigma_t\, dt\, dW_t + \sigma_t^2 (dW_t)^2$$

$$= \sigma_t^2 (dW_t)^2 = \sigma_t^2\, dt$$

$$E(dW_t)^2 = E[W_{t+dt} - W_t]^2 = Var(W_{t+dt} - W_t) = dt$$

- Martingale.

A continuous process X_t is a martingale if for $s<t$,

$$E(X_t \mid \mathcal{F}_s) = X_s$$

Think about a gambling scheme and
X_t is the money at time t

Martingale = fair game

Let X_t satisfy

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t$$

X_t is a martingale if $\mu_t \equiv 0$ for all $t \geq 0$

Example $\quad X_t = x_0 + \mu t + \sigma W_t$

$$dX_t = \mu \, dt + \sigma \, dW_t$$

X_t is martingale if $X_t = x_0 + \sigma W_t$; i.e. $\mu = 0$

$$X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \qquad \text{GBM}$$

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

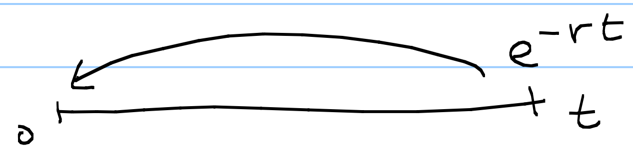
X_t = martingale if $\mu X_t = 0 \iff \mu = 0$.

That is

$$X_t = X_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$$

The present value (discounted value) of stock price

$$Z_t = e^{-rt} S_t \qquad \text{where } S_t \text{ has GBM}$$



Z_t is the price evaluated at time 0

$$Z_t = e^{-rt} S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

$$= S_0 e^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

Risk-neutral valuation $\Rightarrow$ Z_t is a martingale

$\Rightarrow$ $\mu - r = 0$ i.e

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

The product rule   p62

start with two Ito processes

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t$$

$$dY_t = \nu_t \, dt + \rho_t \, dZ_t$$

W_t and Z_t are two standard Brownian motions

The process $X_t Y_t$ has

$$d(X_t Y_t) = X_t \, dY_t + Y_t \, dX_t + (dX_t)(dY_t)$$

Proof

$$
\begin{aligned}
d(X_t Y_t) &= X_{t+dt} Y_{t+dt} - X_t Y_t \\
&= X_{t+dt} Y_{t+dt} - X_{t+dt} Y_t - X_t Y_{t+dt} + X_t Y_t \\
&\quad + X_{t+dt} Y_t + X_t Y_{t+dt} - 2 X_t Y_t \\
&= (X_{t+dt} - X_t)(Y_{t+dt} - Y_t) \\
&\quad + (X_{t+dt} - X_t) Y_t + X_t (Y_{t+dt} - Y_t) \\
&= (dX_t)(dY_t) + (dX_t) Y_t + X_t (dY_t)
\end{aligned}
$$

What is $(dX_t)(dY_t)$

$$(dX_t)(dY_t) = [\mu_t dt + \sigma_t dW_t]\,[\nu_t dt + \rho_t(\rho\, dW_t + \sqrt{1-\rho^2}\,\tilde{W}_t)]$$

$$= \cancel{\mu_t \nu_t (dt)^2} + \cancel{\sigma_t \nu_t\, dt\, dW_t} + \cancel{\mu_t \rho_t \rho\, dt\, dW_t}$$

$$+ \cancel{\mu_t \rho_t \sqrt{1-\rho^2}\, dt\, d\tilde{W}_t}$$

$$+ \cancel{\sigma_t \nu_t\, dt\, dW_t} + \sigma_t \rho_t \rho \underbrace{(dW_t)(dW_t)}_{dt} + \cancel{\sigma_t \rho_t \sqrt{1-\rho^2}\, dW_t\, d\tilde{W}_t}$$

$$E(dW_t\, d\tilde{W}_t) = E(dW_t)\,E(d\tilde{W}_t) = 0 \times 0 = 0$$

$$(dX_t)(dY_t) = \sigma_t \rho_t \rho\, dt$$

$$d(X_t Y_t) = X_t \, dY_t + Y_t \, dX_t + \sigma_t \rho_t \rho \, dt$$

σ_t, ρ_t are two volatilities.

$$\rho = \mathrm{corr}(W_t, Z_t)$$

If $Z_t = W_t$, $\rho = 1$

$$d(X_t Y_t) = X_t \, dY_t + Y_t \, dX_t + \sigma_t \rho_t \, dt$$

If $Z_t = \tilde{W}_t$, $\rho = 0$

$$d(X_t Y_t) = X_t \, dY_t + Y_t \, dX_t$$

P62

#2

#4

Exercise 3.6

S_t stock price with $dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$

$B_t = e^{rt}$, cash bond with $B_0 = 1$

$Z_t = B_t^{-1} S_t = e^{-rt} S_t$ is the PV of S_t

$$dB_t^{-1} = d e^{-rt} = -r e^{-rt} dt = -r B_t^{-1} dt + \rho_t dW_t$$

$\rho_t = 0$

$$dZ_t = B_t^{-1} dS_t + S_t dB_t^{-1} + \sigma_t \rho_t dt$$

$$= B_t^{-1} dS_t + S_t dB_t^{-1}$$

$$= e^{-rt}\left[\mu_t S_t dt + \sigma_t S_t dW_t\right] + S_t \left(-r B_t^{-1} dt\right)$$

$$= e^{-rt}\mu_t S_t\,dt + e^{-rt}\sigma_t S_t\,dW_t - r e^{-rt} S_t\,dt$$

$$= e^{-rt}(\mu_t - r) S_t\,dt + e^{-rt}\sigma_t S_t\,dW_t$$

$$\Rightarrow \quad dZ_t = (\mu_t - r) Z_t\,dt + \sigma_t Z_t\,dW_t \quad !!$$

$\Rightarrow$ Z_t is a martingale iff $\mu_t = r$

$\Rightarrow$ Under the risk-neutral measure

$$dS_t = r S_t\,dt + \sigma_t S_t\,dW_t$$

Let r_t be stochastic, i.e. the interest rate is stochastic

$B_t = e^{\int_0^t r_s ds}$. B_t is the value of the money market account

at t with $B_0 = 1$

$Z_t = B_t^{-1} S_t = e^{-\int_0^t r_s ds} S_t$

Show: $dZ_t = [\mu_t - r_t] Z_t dt + \sigma_t S_t dW_t$

$\Rightarrow$ Under the R-N measure $dS_t = r_t S_t dt + \sigma_t S_t dW_t$

Rules

$dt\, dW_t = 0 \qquad (dt)^2 = 0$

$dt\, d\tilde{W}_t = 0 \qquad dW_t\, d\tilde{W}_t = 0$

$(dW_t)^2 = dt$

3.4 Change of Probability Measure (from P to Q)

Consider a fixed time period $[0, T]$

If there is a positive random variable ξ_T at T

such that $\xi_T > 0$ and $E_P(\xi_T) = 1$ $\quad P =$ Physical probability

We can define a new probablity Q

in the following:

For any random event A,

define $I(A) = \begin{cases} 1, & \text{if } A \text{ happens} \\ 0, & \text{if } A \text{ doesn't happen} \end{cases}$

and Probability of A under new probability Q

is defined as

$$Q(A) = E_P\{ I(A) \xi_T \}$$

Change Probability from P to Q

And $\xi_T = \frac{dQ}{dP}$ called the Radon-Nikodym derivative!

For any Ito process X_t, $0 \le t \le T$

$$E_Q(X_t) = E_P(X_t \xi_T)$$

Let $\xi_t = E_P(\xi_T \mid \mathcal{F}_t)$, then $\xi_t > 0$, $E_P(\xi_t) = 1$

and $E_Q(X_t) = E_P(X_t \xi_t)$.

ξ_t = Radon-Nikodym process
and it is a martingale.

Example

$$\xi_T = e^{-\frac{1}{2}\gamma^2 T + \gamma W_T} > 0 \qquad P71$$

$$E(\xi_T \mid \mathcal{F}_t) = \xi_t = e^{-\frac{1}{2}\gamma^2 t + \gamma W_t} \quad \text{and} \quad E(\xi_t) = 1$$

$e^{-\frac{1}{2}\gamma^2 t + \gamma W_t}$ is a R-N process

Cameron-Martin - Girsanov Theorem

Let W_t be a standard Brownian motion and

γ_t is a process

Define $\tilde{W}_t = \int_0^t \gamma_s \, ds + W_t$

$\tilde{W}_t$ is not a standard Brownian motion under P

There is a Radon-Nikodym derivative

$$\frac{dQ}{dP} = e^{-\int_0^T \gamma_t \, dW_t - \frac{1}{2}\int_0^T \gamma_t^2 \, dt} = \zeta_T > 0, \quad E(\zeta_T) = 1$$

Let $\mathfrak{Z}_t = e^{-\int_0^t \gamma_s dW_s - \frac{1}{2}\int_0^t \gamma_s^2 ds} \Rightarrow$ Show $\mathfrak{Z}_t = E(\mathfrak{Z}_T \mid \mathcal{F}_t)$

such that $\mathbb{Q}$ probability measure obtained from

$\frac{d\mathbb{Q}}{d\mathbb{P}}$ gives that

$\tilde{W}_t$ is a standard Brownian motion

Note that $d\tilde{W}_t = \gamma_t dt + dW_t$

Why do we need the Girsanov Theorem?

Look at

$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$. under $\mathbb{P}$ probability measure

If we want to price an option with underlying S_t

$$dZ_t = d[e^{-rt} S_t]$$

$$= S_t \, d e^{-rt} + e^{-rt} dS_t$$

$$= -re^{-rt} S_t \, dt + e^{-rt} [\mu_t S_t \, dt + \sigma_t S_t \, dW_t]$$

We can NOT use the SDE directly.

This is because $Z_t = e^{-rt} S_t$ satisfies

$$dZ_t = [\mu_t - r] Z_t \, dt + \sigma_t Z_t \, dW_t$$

Z_t is NOT a martingale since $\mu_t - r \neq 0$.

In order to use it for pricing purposes,

Let $d\tilde{W}_t = \gamma_t \, dt + dW_t$ or

$dW_t = -\gamma_t \, dt + d\tilde{W}_t$, γ_t is to be determined!

$$dZ_t = [\mu_t - r] Z_t \, dt + \sigma_t Z_t \left[-\gamma_t \, dt + d\tilde{W}_t \right]$$

$$= [\mu_t - r - \sigma_t \gamma_t] Z_t \, dt + \sigma_t Z_t \, d\tilde{W}_t$$

Z_t is a martingale under Q-probabilitres if and only if $\mu_t - r - \sigma_t \gamma_t \equiv 0 \iff \gamma_t = \dfrac{\mu_t - r}{\sigma_t}$

γ_t = sharpe Ratio.

and $dZ_t = \sigma_t Z_t \, d\tilde{W}_t$.

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_t = \mu_t S_t \, dt + \sigma_t S_t \left[-\frac{\mu_t - r}{\sigma_t} dt + d\tilde{W}_t \right]$$

$$= \mu_t S_t \, dt - (\mu_t - r) S_t \, dt + \sigma_t S_t \, d\tilde{W}_t = r S_t \, dt + \sigma_t S_t \, d\tilde{W}_t,$$

Under the Q probability measure, by the Girsanov,

$\tilde{W}_t$ is a standard Brownian motion.

In practice, change the physical rate of return μ_t to the interest rate r (or r_t) and everything else remains the same! The interest rate can be stochastic!

p 73 – 75

Examples (p75)

1. $X_t = \sigma W_t + \mu t$ under P-probabilities

$$\gamma_t = \frac{\mu}{\sigma} \qquad d\tilde{W}_t = \frac{\mu}{\sigma} dt + dW_t$$

or $$\tilde{W}_t = \frac{\mu}{\sigma} t + W_t$$

Under Q-probabilities,

$$X_t = \sigma\left[-\frac{\mu}{\sigma}t + \tilde{W}_t\right] + \mu t = \sigma \tilde{W}_t$$

X_t is a martingale ($dX_t = \sigma\, d\tilde{W}_t$)

How about $E_P(X_t^2)$ and $E_Q(X_t^2)$

$$E_P(X_t^2) = E_P\{\sigma^2 W_t^2 + 2\mu\sigma t\, W_t + \mu^2 t^2\}$$

$$= \sigma^2 E(W_t^2) + 2\mu\sigma t\, E(W_t) + \mu^2 t^2$$

$$= \sigma^2 t + \mu^2 t^2$$

$$E_Q(X_t^2) = E_Q(\sigma \tilde{W}_t)^2 = \sigma^2 E_Q(\tilde{W}_t^2) = \sigma^2 t$$

$$E_P(X_t^2) \geq E_Q(X_t^2)$$

2. $$dX_t = X_t\left[\sigma dW_t + \mu dt\right] \qquad \text{GBM}$$

Can we find Q-probabilities such that

$$dX_t = X_t\left[\sigma d\tilde{W}_t + \nu dt\right] \quad \text{for } \nu \neq \mu.$$

Yes we can!

$$dW_t = -\gamma_t dt + d\tilde{W}_t$$

$$dX_t = X_t\left[-\sigma\gamma_t dt + \sigma d\tilde{W}_t + \mu dt\right]$$

$$\Rightarrow \quad -\sigma\gamma_t + \mu = \nu \Rightarrow \gamma_t = \frac{\mu - \nu}{\sigma}$$

3.6 Hedging / Replicating strategies

* Self-financing in continuous time

Rebalancing a portfolio without putting in and taking out the money!

Beging with a trading strategy / portfolio (ϕ_t, ψ_t) for a portfolio consisting a stock and the cash bond.

- ϕ_t is the number of shares of S_t at time t
- ψ_t is the number of units of B_t at time t
- The value of the portfolio

$$V_t = \phi_t S_t + \psi_t B_t \qquad B_t = B_0 e^{rt}$$

A self-financing trading strategy is such that

$$dV_t = \phi_t dS_t + \psi_t dB_t \qquad (P81)$$

Think about a very short time period $(t, t+dt]$

LHS = increment of the portfolio value.

RHS = the gains from the 2 securities.

Examples (p81) $r=0$, $B_0=1 \Rightarrow B_t=1$. $S_t=W_t$

(1) $\phi_t=\psi_t=1$, the portfolio $V_t=W_t+1 \equiv \phi_t S_t+\psi_t B_t$

$dV_t = d(W_t+1) = dW_t \overset{?}{=} \phi_t\, dS_t + \psi_t\, dB_t$

RHS $= dS_t + 0 = dW_t$ Yes!

(2) $\phi_t = 2W_t$, $\psi_t = -t - W_t^2$

$V_t = 2W_t \cdot W_t + (-t - W_t^2) = W_t^2 - t$

$dV_t = \phi_t\, dS_t + \psi_t\, dB_t$??

$$dV_t = d[W_t^2] - dt$$

$$= W_t\,dW_t + W_t\,dW_t + (dW_t)^2 - dt$$

$$= 2W_t\,dW_t + dt - dt = 2W_t\,dW_t$$

$\phi_t\,dS_t + \psi_t\,dB_t = 2W_t\,dW_t + 0 = dV_t$ Yes!

$V_t = W_t^2 - t$ is a martingale, (Exercise 3.12)

since no dt term.

Replicating strategy.

For any contingent claim / payoff of an option X_T

if there is a self-financing trading strategy

(ϕ_t, ψ_t) such that $\phi_T S_T + \psi_T B_T = X_T$,

i.e the terminal value of the portfolio is the same as the payoff of the option, the portfolio is called a replicating portfolio and the time-t price of the option is V_t

If the replication is not perfect (no risk) then it is important to perform sensitivity analysis, i.e calculating greeks.

ϕ_t = delta = Δ_t, the sensitivity of the option price to the stock price. $\phi_t = \dfrac{dV_t}{dS_t}$

Other greeks will be introduced later.

3.7 Black-Scholes Model.

p83 One stock and the cash bond

The stock follows a GBM

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The bond has a constant interest rate r

$$dB_t = r B_t dt$$

There is no dividend payments.

Begin with page 87 with $r \geq 0$

Two approaches to prize an option

(i) Risk-neutral Pricing / Valuation

- Find Q-probabilities such that the discounted stock price is a martingale

- The price of a contingent claim / payoff of an option is the expected discounted payoff under Q-probabilities.

The Q-probabilities

$$dS_t = r S_t \, dt + \sigma S_t \, d\tilde{W}_t$$

or $$S_t = S_0 \, e^{(r - \frac{1}{2}\sigma^2) t + \sigma \tilde{W}_t}$$

The discounted stock price

$$Z_t = e^{-rt} S_t \quad \text{has}$$

$$d Z_t = \sigma Z_t \, d\tilde{W}_t$$

Let $\Phi(S_T)$ be the payoff of an option.

The time-t price V_t is

$$V_t = E_Q\left\{ e^{-r(T-t)} \Phi(S_T) \mid \mathcal{F}_t \right\}$$

Let $U_t = e^{-rt} V_t$ be the present value of the option price

Then $U_t = E_Q\left\{ e^{-rT} \Phi(S_T) \mid \mathcal{F}_t \right\}$

U_t is a martingale.

The discounted price of any tradable security is a martingale.

How to calculate an option price?

$$V_t = e^{-r(T-t)} E_Q \{ \Phi(S_T) \mid \mathcal{F}_t \} \qquad \mathcal{F}_t = \{ S_t \}$$

$$= e^{-r(T-t)} E_Q \{ \Phi(S_T) \mid S_t \}$$

$$V_t = V(S_t, t)$$

$$S_T = S_t \, e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma [\tilde{W}_T - \tilde{W}_t]}$$

$$V(S_t, t) = e^{-r(T-t)} E_Q \left\{ \Phi \left(S_t \, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z} \right) \right\}$$

because $\tilde{W}_T - \tilde{W}_t \sim \sqrt{T-t}\, Z$, $Z \sim N(0,1)$

$$V(S_t, t) = e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi\left(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, z}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

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If $Z \sim N(0,1)$, and F, $\bar{\sigma}$ and K are some positive constants,

then

$$\bar{V} = E\left\{\max\left[F e^{\bar{\sigma} Z - \frac{1}{2}\bar{\sigma}^2} - K, 0\right]\right\}$$

$$= F N\left(\frac{\ln\frac{F}{K} + \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\right) - K N\left(\frac{\ln\frac{F}{K} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\right).$$

Derivation

$$E\left\{ \max\left[F e^{\bar{\sigma} Z - \frac{1}{2}\bar{\sigma}^2} - K ,\ 0 \right] \right\}$$

$$= \int_{-\infty}^{\infty} \max\left[F e^{\bar{\sigma} z - \frac{1}{2}\bar{\sigma}^2} - K ,\ 0 \right] \times \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Consider $F e^{\bar{\sigma} z - \frac{1}{2}\bar{\sigma}^2} \geq K$

$$\Rightarrow \quad e^{\bar{\sigma} z - \frac{1}{2}\sigma^2} \geq \frac{K}{F}$$

$$\Rightarrow \quad \bar{\sigma} z - \frac{1}{2}\sigma^2 \geq \ln \frac{K}{F}$$

$$\Rightarrow \quad \bar{\sigma} z \geq \ln \frac{K}{F} + \frac{1}{2} \bar{\sigma}^2$$

$$\Rightarrow \quad z \geq \frac{\ln \frac{K}{F} + \frac{1}{2} \bar{\sigma}^2}{\bar{\sigma}}$$

$$\bar{V} = \int_{\frac{\ln \frac{K}{F} + \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}}^{\infty} \left[F e^{\bar{\sigma} z - \frac{1}{2} \bar{\sigma}^2} - K \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

$$\boxed{z \to -z}$$

$$= \int_{-\infty}^{\frac{\ln \frac{F}{K} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}} \left[F e^{-\bar{\sigma} z - \frac{1}{2} \bar{\sigma}^2} - K \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

$$= F \int_{-\infty}^{\frac{\ln\frac{F}{K} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}} \frac{1}{\sqrt{2\pi}} e^{-\bar{\sigma} z - \frac{1}{2}\sigma^2 - \frac{z^2}{2}} dz$$

$$- K \int_{-\infty}^{\frac{\ln\frac{F}{K} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

The second term is $K N\left(\frac{\ln\frac{F}{K} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\right)$

The first term

$$F \int_{-\infty}^{\frac{\ln \frac{F}{K} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}} \frac{1}{\sqrt{2\pi}} e^{-\bar{\sigma} z - \frac{1}{2}\bar{\sigma}^2 - \frac{z^2}{2}} dz$$

$$= F \int_{-\infty}^{\frac{\ln \frac{F}{K} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[z^2 + 2\bar{\sigma} z + \bar{\sigma}^2\right]} dz$$

$$= F \int_{-\infty}^{\frac{\ln\frac{F}{K} - \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z+\bar{\sigma})^2}{2}} dz$$

$$z + \bar{\sigma} \longrightarrow z$$

$$= F \int_{-\infty}^{\frac{\ln\frac{F}{K} + \frac{1}{2}\bar{\sigma}^2}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = F N\left(\frac{\ln\frac{F}{K} + \frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}} \right)$$

Q.E.D

The Black-Scholes Call Option Pricing Formula

$$\Phi(S_T) = \max[S_T - K, 0]$$

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma[\tilde{W}_T - \tilde{W}_t]}$$

$$\approx S_t e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z}, \quad Z \sim N(0,1)$$

$$V(S_t, t) = e^{-r(T-t)} E_Q\{\max[S_T - K, 0]\}$$

$$= e^{-r(T-t)} E_Q\{\max[S_t e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z}, 0]\}$$

Let $\bar{\sigma} = \sigma\sqrt{T-t}$ $\quad F = S_t e^{r(T-t)}$

$$V(S_t, t) = e^{-r(T-t)}\left[F\,N\left(\frac{\ln\frac{F}{K}+\frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\right) - K\,N\left(\frac{\ln\frac{F}{K}-\frac{1}{2}\bar{\sigma}^2}{\bar{\sigma}}\right)\right]$$

$$= e^{-r(T-t)}\left[S_t e^{r(T-t)} N\left(\frac{\ln\frac{S_t e^{r(T-t)}}{K}+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right.$$

$$\left. - K\,N\left(\frac{\ln\frac{S_t e^{r(T-t)}}{K}-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right]$$

$$= S_t\,N\Bigg(\underbrace{\frac{\ln\frac{S_t}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}_{d_1}\Bigg) - K e^{-r(T-t)} N\Bigg(\underbrace{\frac{\ln\frac{S_t}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}_{d_2 = d_1 - \sigma\sqrt{T-t}}\Bigg)$$

(iii) Replicating Portfolio Approach

For the payoff of an option $\Phi(S_T)$,

if we can form a replicating portfolio (with a self-financing trading strategy (ϕ_t, ψ_t)), then the value of the portfolio at time t is the option price at time t.

Let $V(S_t, t)$ be the value of the replicating portfolio with the self-financing trading strategy (ϕ_t, ψ_t)

$$V(S_t, t) = \phi_t S_t + \psi_t B_t$$

On one hand, we have

$$dV(S_t, t) = \phi_t \, dS_t + \psi_t \, dB_t$$

$$= \phi_t [\mu S_t dt + \sigma S_t dW_t] + \psi_t r B_t dt$$

$$= \mu \phi_t S_t dt + \sigma \phi_t S_t dW_t + r[V(S_t, t) - \phi_t S_t] dt$$

$$= [(\mu - r)\phi_t S_t + r V(S_t, t)] dt + \sigma \phi_t S_t dW_t$$

On the other hand, from Ito's Lemma

$$dV(S_t, t) = [V_t + \frac{1}{2} V_{ss} \sigma^2 S_t^2] dt + V_s \, dS_t$$

$$\boxed{V_t = \frac{\partial V}{\partial t}, \quad V_s = \frac{\partial V}{\partial s}, \quad V_{ss} = \frac{\partial^2 V}{\partial s^2}}$$

$$= \left[V_t + \frac{1}{2} V_{ss} \sigma^2 S_t^2 \right] dt + V_s \left[\mu S_t dt + \sigma S_t dW_t \right]$$

$$= \left[V_t + \mu V_s S_t + \frac{1}{2} V_{ss} \sigma^2 S_t^2 \right] dt + \sigma V_s S_t dW_t$$

Two SDEs must be the same!

$\sigma \phi_t S_t = \sigma V_s S_t$ (equal volatility)

$\Rightarrow \phi_t = V_s$

$(\mu - r) \phi_t S_t + rV = V_t + \mu V_s S_t + \frac{1}{2} V_{ss} \sigma^2 S_t^2$ (equal drift)

$\Rightarrow (\mu - r) V_s S_t + rV = V_t + \mu V_s S_t + \frac{1}{2} \sigma^2 V_{ss} S_t^2$

$$\Rightarrow \quad -r S_t V_s + r V = V_t + \frac{1}{2}\sigma^2 S_t^2 V_{ss}$$

or
$$V_t + r S_t V_s + \frac{1}{2}\sigma^2 S_t^2 V_{ss} = r V$$

The Black-Scholes PDE!!

Since S_t is arbitrary,
$$V_t + r s V_s(s,t) + \frac{1}{2}\sigma^2 s^2 V_{ss}(s,t) = r V(s,t)$$

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If we can solve this PDE, we obtain the replicating portfolio with $\phi_t = V_s$ and $V(s,T) = \Phi(s)$

Check: the B-S call option pring formula

is a solution of the PDE with

$\underline{\Phi}(S) = \max\{S-K, 0\}$

How to hedge an option

General idea: using a replicating portfolio

However, the hedging can not be perfect

in practice; the value of the portfolio is

different from the option price / pay off

Need to minimize the hedging errors by

matching greeks!

Greeks: The sensitivity of the option price to its parameters.

B-S

$$\Delta_t = \frac{\partial V}{\partial S} = \phi_t \qquad N(d_1)$$

$$\Gamma_t = \frac{\partial^2 V}{\partial S^2} = V_{SS} \qquad \frac{n(d_1)}{S_t \sigma \sqrt{T-t}} \qquad n(x) = N'(x)$$

$$\theta_t = \frac{\partial^2 V}{\partial t} = V_t \qquad -\frac{S_t\, n(d_1)\, \sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)} N(d_2)$$

$$\rho_t = \frac{\partial V}{\partial r} = \text{duration} \qquad K(T-t)e^{-r(T-t)} N(d_2)$$

$$\text{Vega} = \frac{\partial V}{\partial \sigma} \qquad S_t\, n(d_1) \sqrt{T-t}$$

Exercise 3.17 p 98

A stock has current price \$10 and follows a GBM.

$\mu = 15\%$, $\sigma = 20\%$, $r = 5\%$.

What is the value of a derivative that pays off \$1 if the stock prize is more that \$10 in a year's time?

Under Q-probabilities

$$S_t = 10\, e^{(0.05 - \frac{1}{2} \times 0.2^2) t + 0.2 \tilde{W}_t}$$

$$S_t = 10\, e^{0.03t + 0.2\tilde{W}_t}$$

$$V(S_0, 0) = e^{-rT} E_Q\left(\Phi(S_T)\right)$$

$$\Phi(S_1) = \begin{cases} 1, & S_1 > 10 \\ 0, & S_1 \le 10 \end{cases}$$

$$S_1 = 10\, e^{0.03 + 0.2Z} \qquad Z \sim N(0,1)$$

$$S_1 > 10 \Rightarrow e^{0.03 + 0.2Z} > 1$$

$$\Rightarrow 0.03 + 0.2Z > 0 \Rightarrow Z > -\frac{0.03}{0.2} = -\frac{0.3}{2}$$

$$= -0.15$$

$$V(10,0) = e^{-0.05} \int_{-0.15}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = e^{-0.05} N(0.15)$$

How do you hedge this option?

Need to calculate Δ_0

Change the current price from \$10 to \$10.1

Recalculate the option price

$V(10.1, 0)$

$$\Delta_0 \doteq \frac{V(10.1, 0) - V(10, 0)}{10.1 - 10}$$